Reg. No. : ....

Code No. : 5089 **Sub. Code : HMAM 42**

M.Sc. (CBCS) DEGREE EXAMINATION,
APRIL 2020.

Fourth Semester

Mathematics

MEASURE AND INTEGRATION

(For those who joined in July 2012-2015)

Time : Three hours Maximum : 75 marks

PART A — (10 × 1 = 10 marks)

Answer ALL questions.

Choose the correct answer :

1. If A is measurable and B is any set disjoint from A, then

   (a) $m^*(A \cup B) = m^*(A) + m^*(B)$

   (b) $m^*(A \cup B) = m^*(A)$

   (c) $m^*(A \cup B) = m^*(B)$

   (d) $m^*(A \cup B) = m^*(A) + m^*(B) - m^*(A \cap B)$

2. If E_1 and E_2 are measurable, then

(a) $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

(b) $m(E_1 \cup E_2) = m(E_1) + m(E_2) + m(E_1 \cap E_2)$

(c) $m(E_1 \cup E_2) = m(E_1) + m(E_2)$

(d) $m(E_1 \cup E_2) = m(E_1) - m(E_2) + m(E_1 \cap E_2)$

3. Let the function f have a measurable domain E.

Then which one of the following is true

(a) $\{x \in E / f(x) = \infty\} = \bigcap_{K=1}^{\infty} \{x \in E / f(x) \geq K\}$

(b) $\{x \in E / f(x) = \infty\} = \bigcap_{K=1}^{\infty} \{x \in E / f(x) > K\}$

(c) $\{x \in E / f(x) = \infty\} = \bigcap_{K=1}^{\infty} \{x \in E / f(x) < K\}$

(d) $\{x \in E / f(x) = \infty\} = \bigcap_{K=1}^{\infty} \{x \in E / f(x) \leq K\}$

4. Let f be a bounded measurable function on a set of finite measure E. Suppose A and B are disjoint measurable subsets on E. Then

(a) $\int_{A\cup B} f = \int_A f + \int_B f - \int_{A\cap B} f$

(b) $\int_{A\cup B} f = \int_A f + \int_B f + \int_{A\cap B} f$

(c) $\int_{A\cup B} f = \int_A f + \int_B f$

(d) $\int_{A\cup B} f = \int_A f - \int_B f$

5. The function f defined on $[0,1]$ by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & for \;\; 0 < x \leq 1 \\ 0 & for \;\; x = 0 \end{cases}$, then

(a) f' is integrable over $[0,1]$

(b) f' is not integrable over $[0,1]$

(c) not continuous

(d) none of these

6. The functions f and g on $[-1,1]$ by $f(x)=x^{1/3}$ for $-1\le x\le 1$ and

$$g(x)=\begin{cases} x^2\cos(\pi/2x) \; if \;\; x=0, x\in[-1,1] \\ 0 \qquad\qquad\quad if \;\; x=0 \end{cases}$$

(a) neither f and nor g are absolutely continuous on $[-1,1]$

(b) f is absolutely continuous on $[-1,1]$ and g is not absolutely continuous

(c) both f and g are absolutely continuous $[-1,1]$

(d) none of these

7. If A and B are measurable sets and $A\le B$, then

(a) $\mu(A)<\mu(B)$

(b) $\mu(A)=\mu(B)$

(c) $\mu(A)\le\mu(B)$

(d) $\mu(A)\ge\mu(B)$

8. For an outer measure $\mu^*:2^X\to[0,\infty]$, we call a subset E of X measurable (with respect to μ^*) provided for every subset A of X,

(a) $\mu^*(A)=\mu^*(A\cap E)$

(b) $\mu^*(A)=\mu^*(A\cap E^C)$

(c) $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^C)$

(d) $\mu^*(A)=\mu^*(A\cup E)+\mu^*(A\cup E^C)$

9. Consider a $\sigma-$ algebra $M = \{X, \phi\}$. Then the only measurable functions are

(a) Bounded functions

(b) Constant functions

(c) Continuous functions

(d) None of these

10. Let (X, M, μ) be a measure space and f a non negative measurable function on X for which $\int_X f \, d\mu < \infty$, then one of them is true

(a) $\{x \in X / f(x) > 0\}$ is a σ–finite

(b) $\{x \in X / f(x) > 0\}$ is not a σ–finite

(c) $\{x \in X / f(x) > 0\}$ is finite

(d) None of these

PART B — (5 × 5 = 25 marks)

Answer ALL questions, choosing either (a) or (b).

11. (a) Prove that the union of a countable collection of measurable sets is measurable.

Or

(b) Let f and g be measurable functions on E that are finite almost everywhere on E. Then prove that fg is measurable on E.

12. (a) Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise almost everywhere on E to the function f. Then prove that f is measurable.

Or

(b) State and prove bounded convergence theorem.

13. (a) Let f be an increasing function on the closed bounded interval $[a, b]$. Then prove that for each $\alpha > 0$,

$$m^* \{x \in (a, b) / \overline{D} f(x) \geq \alpha\} \leq \frac{1}{\alpha} . [f(b) - f(a)]$$ and

$$m^* \{x \in (a, b) / \overline{D} f(x) = \infty\} = 0.$$

Or

(b) Prove that a function f on a closed bounded interval $[a, b]$ is absolutely continuous on $[a, b]$ if and only if it is an indefinite integral over $[a, b]$.

14. (a) Let γ be a signed measure on the measurable space (X, M). Prove that every measurable subset of a positive set is itself positive and the union of a countable collection of positive sets is positive.

Or

(b) Prove that the union of a finite collection of measurable sets is measurable.

15. (a) Let (X, M) be a measurable space, f a measurable real-valued function on X, and $\phi : R \to R$ continuous. Then prove that the composition $\phi \circ f : X \to R$ also is measurable.

Or

(b) Let (X, M, μ) be a measure space and f a nonnegative measurable function on X for which $\int_X f \, d\mu < \infty$. Then prove that f is finite almost everywhere on X and $\{x \in X / f(x) > 0\}$ is $\sigma -$finite.

PART C — (5 × 8 = 40 marks)

Answer ALL questions, choosing either (a) or (b).

16. (a) Prove that the outer measure of an interval is its length.

Or

(b) Prove that the Lebesque measure possesses the following continuity properties

(i) If $\{A_K\}_{K=1}^{\infty}$ is an ascending collection of measurable sets, then

$$m\left(\bigcup_{K=1}^{\infty} A_k\right) = \lim_{K\to\infty} m(A_k)$$

(ii) If $\{B_K\}_{K=1}^{\infty}$ is a descending collection of measurable sets and $m(B_1) < \infty$, then

$$m\left(\bigcap_{K=1}^{\infty} B_K\right) = \lim_{K\to\infty} m(B_K).$$

17. (a) Assume E has finite measure. Let $\{f_n\}$ be a sequence of measurable functions on E that converges pointwise on E to the real valued function f. Then prove that for each $\in > 0$, there is a closed set F contained in E for which $\{f_n\} \to f$ uniformly on F and $m(E \sim F) < \in$.

Or

(b) State and prove the Lebesgue dominated convergence theorem.

18. (a) Prove that if the function f is monotone on the open interval (a, b), then it is differentiable almost everywhere on (a, b).

Or

(b) Prove that a function f is of bounded variation on the closed bounded interval $[a, b]$ if and only if it is the difference of two increasing functions on $[a, b]$.

19. (a) State and prove the Hahn Decomposition theorem.

Or

(b) Prove that the union of a countable collection of measurable sets is measurable.

20. (a) State and prove Fatou's lemma.

Or

(b) Let (X, M, μ) be a measure space and $\{f_n\}$ a sequence of functions on X that is both uniformly integrable and tight over X. Assume $\{f_n\} \to f$ pointwise almost everywhere on X and the function f is integrable over X. Then prove that $\lim_{n\to\infty} \int_E f_n \, d\mu = \int_E f \, d\mu$.